

THE ADDITIVE STRUCTURE OF CARTESIAN PRODUCTS SPANNING FEW DISTINCT DISTANCES

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ABSTRACT. Guth and Katz proved that any point set \mathcal{P} in the plane determines $\Omega(|\mathcal{P}|/\log|\mathcal{P}|)$ distinct distances. We show that when near to this lower bound, a point set \mathcal{P} of the form $A \times A$ must satisfy $|A - A| \ll |A|^{2-1/8}$.

1. INTRODUCTION

If \mathcal{P} is a set of N points in the plane, let $\Delta(\mathcal{P})$ denote the set of (squared) distances spanned by \mathcal{P} , that is

$$\Delta(\mathcal{P}) = \{(p_1 - q_1)^2 + (p_2 - q_2)^2 : (p_1, p_2), (q_1, q_2) \in \mathcal{P}\}.$$

Erdős famously conjectured [E1] that any set of points needs to determine at least a $\Omega(N/\sqrt{\log N})$ distinct numbers as distances. This lower bound occurs when the points are arranged in a square grid in the integer lattice, for instance. In a landmark paper [GK], Guth and Katz very nearly established Erdős' conjecture, by proving the lower bound:

Theorem 1 (Guth-Katz). *If \mathcal{P} is a set of N points in the plane then $|\Delta(\mathcal{P})| \gg \frac{N}{\log N}$.*

Though the problem of estimating the number of distinct distances is now almost resolved, what remains to be shown is a characterization of those point sets for which the lower bound of Theorem 1 is sharp. In this note, we think of an extremal configuration as a set of points \mathcal{P} spanning $O(|\mathcal{P}|^{1+\epsilon})$ distances. The known examples of such sets all appear to come from sets with algebraic structure - in the grid example above, the points are coming from a lattice. Another example is to take the vertices of a regular n -gon, or n equally spaced points on a line. One might ask if this algebraic structure is a necessary feature

of such point sets - in the listed examples, a very strong rotational or translational symmetry is present. Such a conjecture was made by Erdős in [E2], where he states that all extremal configurations should exhibit a lattice-like structure. Some progress towards such a result is given in [SZZ]. Erdős is pretty vague about what he means by lattice like, but we are going to take a lattice to mean a discrete subgroup of the plane. Here, we will prove a theorem in further support of this fact. Let A be a set of real numbers such that $\mathcal{P} = A \times A$ determines few distances. Theorem 1 says that there are at least $c|A|^2/\log |A|$ distinct distances for some positive, absolute constant c . We will show that when near this lower bound, the difference set

$$D = A - A = \{a_1 - a_2 : a_1, a_2 \in A\}$$

has to be somewhat small, thus showing that the set \mathcal{P} is to some extent additively structured. It can certainly be argued that the assumption that \mathcal{P} is a cartesian product means that our point set is already lattice-like, but I think that the additive behaviour of A is essential to be considered truly lattice-like.

Theorem (Main theorem). *Suppose A is a finite set of real numbers and let $\Delta(A \times A)$ be the set of distances spanned by $A \times A$. Then*

$$|A - A| \ll |\Delta(A \times A)| |A|^{-1/8}.$$

In [E2], it was conjectured that in extremal point configurations there should be many points ($|\mathcal{P}|^{1/2}$ is conjectured, but $|\mathcal{P}|^\epsilon$ is already interesting) on a line or circle. In the case of a cartesian product, there are trivially many points on axis parallel lines, or the main diagonal. However, this theorem also shows that there are non-trivial lines which contain many points.

Theorem (Rich lines). *Suppose A is a finite set of real numbers such that $\Delta(A \times A) = O(|A|^2)$. There is a non-trivial line (in fact many) which contains $\Omega(|A|^{1/8})$ points of $A \times A$.*

Proof. Since

$$|A|^2 = \sum_{d \in A - A} \sum_{\substack{a, b \in A \\ a - b = d}} 1,$$

one of the inner sums is at least $\Omega(|A|^{1/8})$. Thus there are $\Omega(|A|^{1/8})$ points $(a, b) \in A \times A$ with $a - b = d$, and this equation defines the desired line. \square

From an arithmetic combinatorial point of view, our main theorem ought to be true because the set of distances spanned by \mathcal{P} is the set of numbers $\Delta = D^2 + D^2$, where $D^2 = \{d^2 : d \in D\}$. Heuristically, the squaring of an additively structured set (in this case D , a difference set) should result in a set (in this case D^2) which is not additively structured. The fact that squaring, or in fact any convex function, meddles with the additive properties of a set of numbers is captured in a theorem of Elekes-Nathanson-Ruzsa from [ENR], later improved by Li and Roche-Newton [LRN].

Theorem 2 (Convexity and Sumsets). *Let S be a finite set of real numbers, let f be a strictly convex function and let $\varepsilon > 0$. Then*

$$\max\{|S + S|, |f(S) + f(S)|\} \gg |S|^{14/11-\varepsilon}.$$

In particular,

$$\max\{|S + S|, |S^2 + S^2|\} \gg |S|^{14/11-\varepsilon}.$$

Viewed in this way, the main theorem is really one in arithmetic combinatorics. Recently, Shkredov has proved similar theorems, examining the multiplicative structure (or lack thereof) of difference sets, see [S].

We close this introduction by remarking that if a well-known conjecture of Rudin holds (this form is due to Ruzsa, [CG]), then a very strong improvement of the main theorem can be made to points with integral co-ordinates.

Conjecture (Rudin/Ruzsa). *For $\varepsilon > 0$, if S is a set of perfect squares then $|S + S| \gg_\varepsilon |S|^{2-\varepsilon}$.*

Since $D = A - A \subset \mathbb{Z}$, D^2 is a set of perfect squares. It follows that $\Delta(A \times A) \gg_\varepsilon |D|^{2-\varepsilon}$. If $\Delta(A \times A) = O(|A|^2)$ then $|D| = O(|A|^{1+\delta})$ for some $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$.

2. PROOFS

The fundamental observation in this paper is a simple one. By taking particular elements of

$$2D^2 - 2D^2 = \{d_1^2 + d_2^2 - d_3^2 - d_4^2 : d_1, d_2, d_3, d_4 \in D\}$$

we can find a dilated copy of $D \cdot D$.

Lemma 1. *If D is a difference set then $2D^2 - 2D^2$ contains a dilate of the set $D \cdot D$.*

Proof. Let $D = A - A$ and suppose $d_1 = a_2 - a_1$ and $d_2 = b_1 - b_2 \in D$. Then

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 - (b_1 - a_2)^2 - (b_2 - a_1)^2 = 2d_1d_2.$$

The left hand side clearly shows this element of $2 \cdot D \cdot D$ belongs to $2D^2 - 2D^2$. \square

The other essential ingredient is the Plunnecke-Ruzsa Theorem, which is nicely proved in [P].

Theorem 3 (Plunnecke-Ruzsa). *Suppose S is a finite subset of an abelian group. Then*

$$|mS - nS| \leq \left(\frac{|S + S|}{|S|} \right)^{n+m} |S|.$$

Combining these facts gives a number of options, and I am certain that there is a more efficient idea that could lead to a quantitative improvement of the main theorem. In particular, avoiding such a large exponent in the use of Theorem 3 could drastically improve the final result. In a first iteration of this paper, we would have used that $2D^2 - 2D^2$ contains a dilate of D . Then because of Theorem 2, $4D^2 - 4D^2$ (which contains a dilate of $D + D$) is bounded from below in terms of $|D|$. We can get a better exponent by instead using an idea of Solymosi from his well-known work on the Sum-Product problem, [So].

Lemma 2. *Let \mathcal{L} be a collection of L lines in the plane which pass through the origin. Let \mathcal{P} be a collection of points lying in a single*

quadrant in the plane, and such that each line $l \in \mathcal{L}$ contains at least n points from \mathcal{P} . Then

$$|\mathcal{P} + \mathcal{P}| \geq (L - 1)n^2.$$

Proof. Sort the lines in \mathcal{L} by increasing slope. Let l_1 and l_2 be adjacent lines. Then the (vector) sum of any point on l_1 and any point on l_2 lies between l_1 and l_2 . This means that all sums of points in \mathcal{P} coming from l_1 and l_2 are distinct from the sums coming from any other two adjacent lines in \mathcal{L} . Moreover, all such sums are distinct by linear independence, so any two adjacent lines produce n^2 sums. Since there are $L - 1$ adjacent pairs, the lemma follows. \square

Corollary 1. *Let S be a set of real numbers and let $S/S = \{s_1/s_2 : s_1, s_2 \in S\}$. Then*

$$|S \cdot S + S \cdot S|^2 \gg |S/S||S|^2.$$

Proof. We will apply Lemma 2 to the set

$$\mathcal{P} = \{(s_1s, s_2s) : s_1, s_2, s \in S\},$$

which is a subset of $(S \cdot S) \times (S \cdot S)$. Let $r = s_1/s_2 \in S/S$. Choosing $s \in S$ arbitrarily, we have $|S|$ points (s_2s, s_1s) , and each lies on the line l_r through the origin with slope r . Let \mathcal{L}_1 be the set of such lines which pass through the first and third quadrants, and \mathcal{L}_2 the set of those that pass through the second and fourth quadrants. Then one of $|\mathcal{L}_1|$ or $|\mathcal{L}_2|$ has size at least $\frac{1}{2}|S/S|$. Assume $|\mathcal{L}_1| \geq \frac{1}{2}|S/S|$ as the other case is similar. Next, let \mathcal{L}_1^+ denote those lines which have at least half of their points from $(S \cdot S) \times (S \cdot S)$ in the first quadrant and \mathcal{L}_1^- those with at least half of their points in the third quadrant. One of the two sets has at least $\frac{1}{4}|S/S|$ lines in it, and each line has at least $|S|/2$ points in some fixed quadrant. Now the corollary follows from Lemma 2, the fact that

$$\mathcal{P} + \mathcal{P} \subset (S \cdot S) \times (S \cdot S) + (S \cdot S) \times (S \cdot S),$$

and the fact that

$$(S \cdot S \times S \cdot S) + (S \cdot S \times S \cdot S) = (S \cdot S + S \cdot S) \times (S \cdot S + S \cdot S).$$

\square

Finally, we recall a beautiful theorem due to Ungar ([U]), which gives a lower bound for the number of slopes defined by a set of points.

Theorem 4. *Let \mathcal{P} be a finite set of points, not all on a line. Then the set*

$$\left\{ \frac{y_1 - y_2}{x_1 - x_2} : (x_1, y_1), (x_2, y_2) \in \mathcal{P} \right\}$$

has size at least $|\mathcal{P}| - 1$.

Proof of Main theorem. Set $D = A - A$ and $\Delta = \Delta(\mathcal{P}) = D^2 + D^2$. From Lemma 1 and Theorem 3, we see that

$$|D \cdot D + D \cdot D| \leq |4D^2 - 4D^2| \leq \left(\frac{|D^2 + D^2|}{|D^2|} \right)^8 |D^2| = \frac{|\Delta|^8}{|D|^7}.$$

From Corollary 1, we have that

$$|D \cdot D + D \cdot D| \gg |D| |D/D|^{1/2}.$$

But D/D is the set of slopes from the set $A \times A$, so Theorem 4 gives

$$|D \cdot D + D \cdot D| \gg |D| |A|.$$

Combining these estimates gives that

$$|D| \ll |\Delta| |A|^{-1/8}.$$

The theorem follows. □

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